3 Triangles

Let's study.

- Theorem of remote interior angles of a triangle
- Congruence of triangles
- Theorem of an isosceles triangle
- Property of 30°- 60°- 90° angled triangle
- Median of a triangle
- Property of median on hypotenuse of a right angled triangle
- Perpendicular bisector theorem
- Angle bisector theorem
- Similar triangles

Activity :

Draw a triangle of any measure on a thick paper. Take a point T on ray QR as shown in fig. 3.1. Cut two pieces of thick paper which will exactly fit the corners of $\angle P$ and $\angle Q$. See that the same two pieces fit exactly at the corner of $\angle PRT$ as shown in the figure.

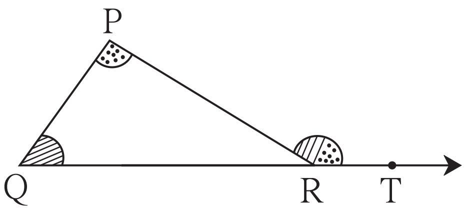


Fig. 3.1

Let's learn.

Theorem of remote interior angles of a triangle

Theorem : The measure of an exterior angle of a triangle is equal to the sum of its remote interior angles.

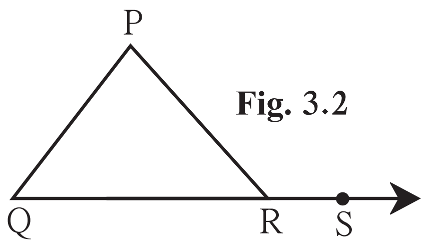


Fig. 3.2

Given : $\angle PRS$ is an exterior angle of Δ PQR.

To prove : $\angle PRS = \angle PQR + \angle QPR$

Proof : The sum of all angles of a triangle is 180°.

$\therefore$ $\angle PQR + \angle QPR + \angle PRQ = 180^\circ$.......(I)

$\angle PRQ + \angle PRS = 180^\circ$.....angles in linear pair......(II)

$\therefore$ from (I) and (II)

$\angle PQR + \angle QPR + \angle PRQ = \angle PRQ + \angle PRS$

$\therefore$ $\angle PQR + \angle QPR = \angle PRS$ ....eliminating $\angle PRQ$ from both sides

$\therefore$ the measure of an exterior angle of a triangle is equal to the sum of its remote interior angles.

Use your brain power! Can we give an alternative proof of the theorem drawing a line through point R and parallel to seg PQ in figure 3.2 ?

Let's learn.

Property of an exterior angle of triangle

The sum of two positive numbers a and b, that is $(a + b)$ is greater than a and greater than b also. That is, $a + b > a$, $a + b > b$

Using this inequality we get one property relaed to exterior angle of a triangle.

If $\angle PRS$ is an exterior angle of Δ PQR then

$\angle PRS > \angle P$, $\angle PRS > \angle Q$

$\therefore$ an exterior angle of a triangle is greater than its remote interior angle.

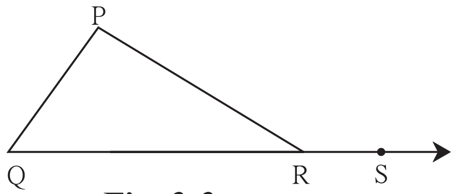


Fig. 3.3

Solved examples

Ex (1) The measures of angles of a triangle are in the ratio 5 **:** 6 **:** 7. Find the measures.

Solution : Let the measures of the angles of a triangle be $5x$, $6x$, $7x$.

$$\therefore \quad 5x + 6x + 7x = 180^\circ$$

$$18x = 180^\circ$$

$$x = 10^\circ$$

$5x = 5 \times 10 = 50^\circ$, $6x = 6 \times 10 = 60^\circ$, $7x = 7 \times 10 = 70^\circ$

$\therefore$ the measures of angles of the triangle are 50°, 60° and 70°.

Ex (2) Observe figure 3.4 and find the measures of $\angle PRS$ and $\angle RTS$.

Solution : $\angle PRS$ is an exterior angle of Δ PQR.

So from the theorem of remote interior angles,

$\angle PRS = \angle PQR + \angle QPR$

$= 40^\circ + 30^\circ$

$= 70^\circ$

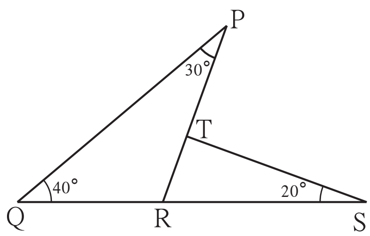


Fig. 3.4

In Δ RTS

$\angle TRS + \angle RTS + \angle TSR = \square$ .... sum of all angles of a triangle

$\therefore \square + \angle RTS + \square = 180^\circ$

$\therefore \angle RTS + 90^\circ = 180^\circ$

$\therefore \angle RTS = \square$

Ex (3) Prove that the sum of exterior angles of a triangle, obtained by extending its sides in the same direction is 360°.

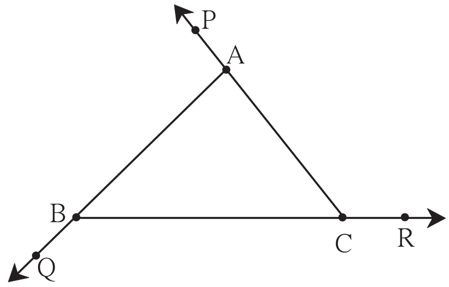


Fig. 3.5

Given : $\angle PAB$, $\angle QBC$ and $\angle ACR$ are exterior angles of Δ ABC

To prove : $\angle PAB + \angle QBC + \angle ACR = 360^\circ$

Proof : **Method I**

Considering exterior $\angle PAB$ of Δ ABC,

$\angle ABC$ and $\angle ACB$ are its remote interior angles.

$\angle PAB = \angle ABC + \angle ACB$ ----(I)

Similarly, $\angle ACR = \angle ABC + \angle BAC$ ----(II)..theorem of remote interior angles

and $\angle CBQ = \angle BAC + \angle ACB$ ---- (III)

Adding (I), (II) and (III),

$\angle PAB + \angle ACR + \angle CBQ$

$= \angle ABC + \angle ACB + \angle ABC + \angle BAC + \angle BAC + \angle ACB$

$= 2\angle ABC + 2\angle ACB + 2\angle BAC$

$= 2\,(\angle ABC + \angle ACB + \angle BAC)$

$= 2 \times 180^\circ$ . . . . . sum of interior angles of a triangle

$= 360^\circ$

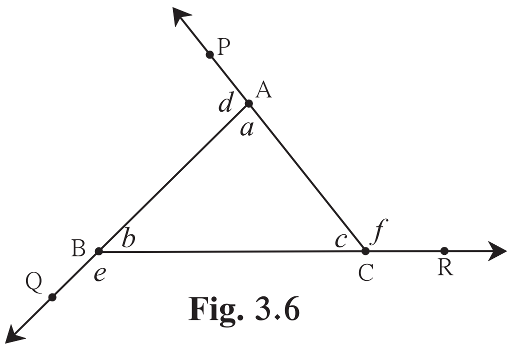


Fig. 3.6

Method II

$\angle c + \angle f = 180^\circ$ . . . . (angles in linear pair)

Also, $\angle a + \angle d = 180^\circ$

and $\angle b + \angle e = 180^\circ$

$\therefore \angle c + \angle f + \angle a + \angle d + \angle b + \angle e = 180^\circ \times 3 = 540^\circ$

$\angle f + \angle d + \angle e + (\angle a + \angle b + \angle c) = 540^\circ$

$\therefore \angle f + \angle d + \angle e + 180^\circ = 540^\circ$

$\therefore f + d + e = 540^\circ - 180^\circ$

$= 360^\circ$

Ex (4) In figure 3.7, bisectors of $\angle B$ and $\angle C$ of Δ ABC intersect at point P.

Prove that $\angle BPC = 90 + \frac{1}{2} \angle BAC$.

Complete the proof filling in the blanks.

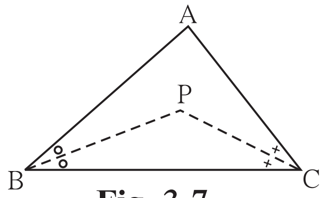


Fig. 3.7

Proof : In Δ ABC,

$\angle BAC + \angle ABC + \angle ACB = \square$ ...... sum of measures of angles of a triangle

$\therefore \frac{1}{2} \angle BAC + \frac{1}{2} \angle ABC + \frac{1}{2} \angle ACB = \frac{1}{2} \times \square$

....multiplying each term by $\frac{1}{2}$

$\therefore \frac{1}{2} \angle BAC + \angle PBC + \angle PCB = 90^\circ$

$\therefore \angle PBC + \angle PCB = 90^\circ - \frac{1}{2} \angle BAC$ ......(I)

In Δ BPC

$\angle BPC + \angle PBC + \angle PCB = 180^\circ$ ......sum of measures of angles of a triangle

$\therefore \angle BPC + \square = 180^\circ$ ......from (I)

$\therefore \angle BPC = 180^\circ - (90^\circ - \frac{1}{2} \angle BAC)$

$= 180^\circ - 90^\circ + \frac{1}{2} \angle BAC$

$= 90^\circ + \frac{1}{2} \angle BAC$

Practice set 3.1

1. In figure 3.8, $\angle ACD$ is an exterior angle of Δ ABC. $\angle B = 40^\circ$, $\angle A = 70^\circ$. Find the measure of $\angle ACD$.

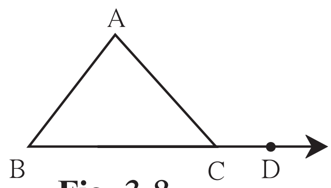


Fig. 3.8

2. In Δ PQR, $\angle P = 70^\circ$, $\angle Q = 65^\circ$ then find $\angle R$.

3. The measures of angles of a triangle are x°, $(x-20)^\circ$, $(x-40)^\circ$. Find the measure of each angle.

4. The measure of one of the angles of a triangle is twice the measure of its smallest angle and the measure of the other is thrice the measure of the smallest angle. Find the measures of the three angles.

5. In figure 3.9, measures of some angles are given. Using the measures find the values of x, y, z.

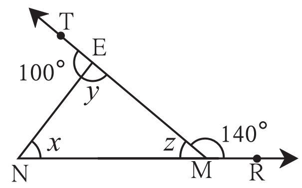


Fig. 3.9

6. In figure 3.10, line AB || line DE. Find the measures of $\angle$DRE and $\angle$ARE using given measures of some angles.

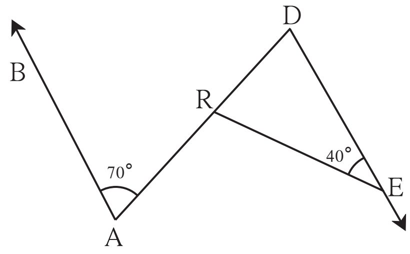


Fig. 3.10

7. In Δ ABC, bisectors of $\angle$A and $\angle$B intersect at point O. If $\angle C = 70°$. Find measure of $\angle$AOB.

8. In Figure 3.11, line AB || line CD and line PQ is the transversal. Ray PT and ray QT are bisectors of $\angle$BPQ and $\angle$PQD respectively.
Prove that m$\angle PTQ = 90°$.

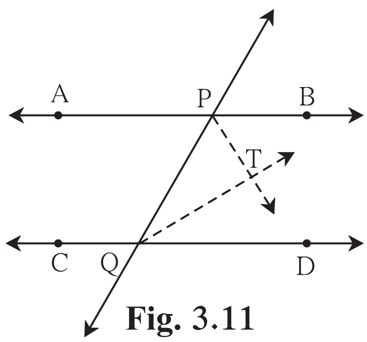


Fig. 3.11

9. Using the information in figure 3.12, find the measures of $\angle a$, $\angle b$ and $\angle c$.

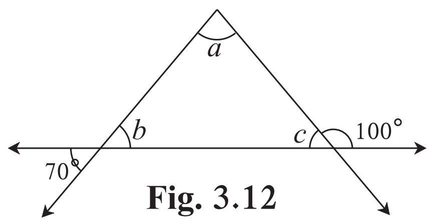


Fig. 3.12

10. In figure 3.13, line DE || line GF ray EG and ray FG are bisectors of $\angle$DEF and $\angle$DFM respectively.
Prove that,
(i) $\angle DEG = \frac{1}{2} \angle EDF$ (ii) EF = FG.

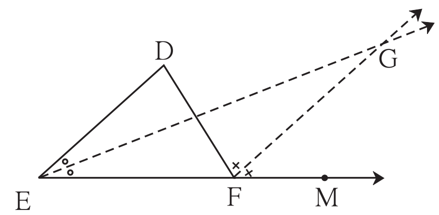


Fig. 3.13

Let's learn.

Congruence of triangles

We know that, if a segment placed upon another fits with it exactly then the two segmetns are congruent. When an angle placed upon another fits with it exactly then the two angles are congruent. Similarly, if a triangle placed upon another triangle fits exactly with it then the two triangles are said to be congruent. If Δ ABC and Δ PQR are congruent is written as $\Delta ABC \cong \Delta PQR$.

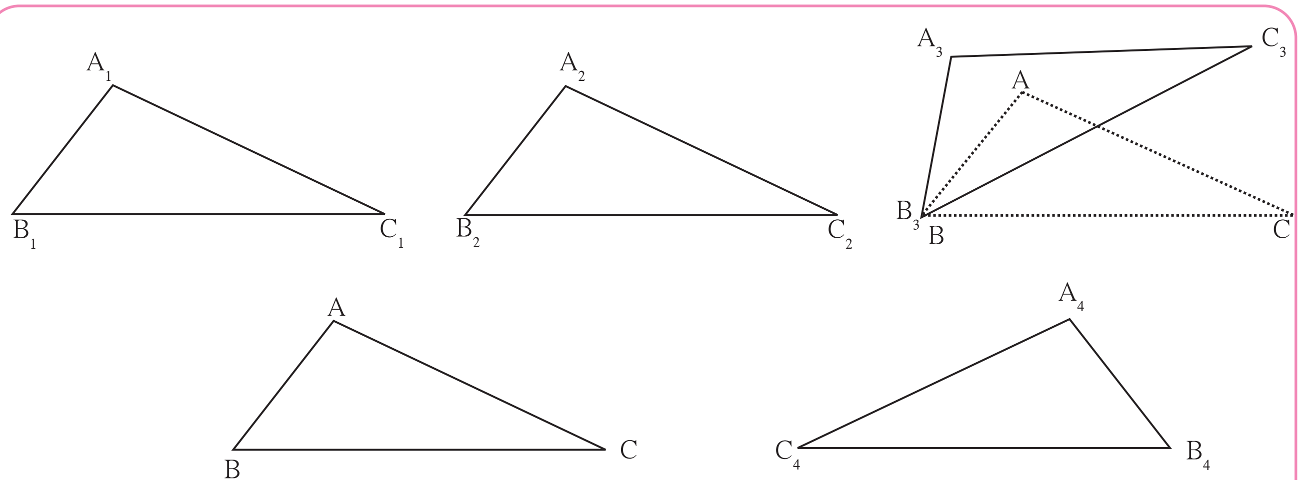


Fig. 3.14

Activity : Draw Δ ABC of any measure on a card-sheet and cut it out.

Place it on a card-sheet. Make a copy of it by drawing its border. Name it as $\Delta A_1B_1C_1$

Now slide the Δ ABC which is the cut out of a triangle to some distance and make one more copy of it. Name it $\Delta A_2B_2C_2$.

Then rotate the cut out of triangle ABC a little, as shown in the figure, and make another copy of it. Name the copy as $\Delta A_3B_3C_3$. Then flip the triangle ABC, place it on another card-sheet and make a new copy of it. Name this copy as $\Delta A_4B_4C_4$.

Have you noticed that each of $\Delta A_1B_1C_1$, $\Delta A_2B_2C_2$, $\Delta A_3B_3C_3$ and $\Delta A_4B_4C_4$ is congruent with Δ ABC ? Because each of them fits exactly with Δ ABC.

Let us verify for $\Delta A_3B_3C_3$. If we place $\angle A$ upon $\angle A_3$, $\angle B$ upon $\angle B_3$ and $\angle C$ upon $\angle C_3$, then only they will fit each other and we can say that $\Delta ABC \cong \Delta A_3B_3C_3$.

We also have $AB = A_3B_3$, $BC = B_3C_3$, $CA = C_3A_3$.

Note that, while examining the congruence of two triangles, we have to write their angles and sides in a specific order, that is with a specific one-to-one correspondence.

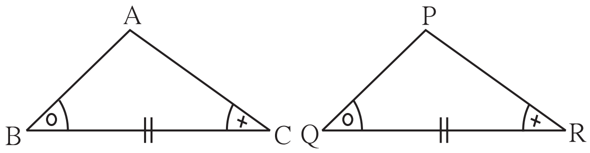
If $\Delta ABC \cong \Delta PQR$, then we get the following six equations :

$\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ . ... (I) and $AB = PQ, BC = QR, CA = RP$ . . . . (II)

This means, with a one-to-one correspondence between the angles and the sides of two triangles, we get three pairs of congruent angles and three pairs of congruent sides.

Given six equations above are true for congruent triangles. For this let us see three specific equations are true then all six equations become true and hence two triangles congruent.

(1) In a correspondence, if two angles of ΔABC are equal to two angles of ΔPQR and the sides included by the respective pairs of angles are also equal, then the two triangles are congruent.



This property is called as angle-side-angle test, which in short we write A-S-A test.

Fig. 3.15

(2) In a correspondence, if two sides of Δ ABC are equal to two sides of Δ PQR and the angles included by the respective pairs of sides are also equal, then the two triangles are congruent.

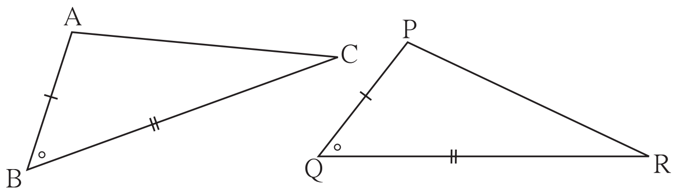


Fig. 3.16

This property is called as side-angle-side test, which in short we write S-A-S test.

(3) In a correspondence, if three sides of Δ ABC are equal to three sides of Δ PQR , then the two triangles are congruent.

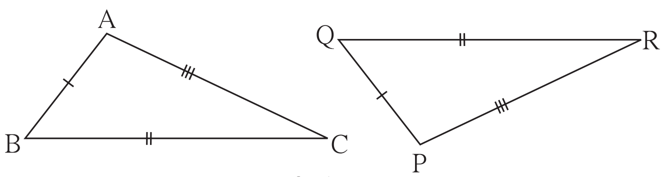
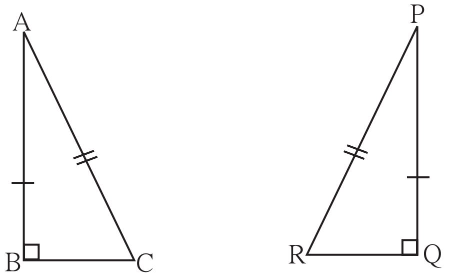


Fig. 3.17

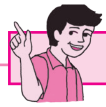
This property is called as side-side-side test, which in short we write S-S-S test.

(4) If in Δ ABC and Δ PQR, $\angle$B and $\angle$Q are right angles, hypotenuses are equal and AB = PQ, then the two triangles are congruent.



This property is called the hypotenuse side test.

Fig. 3.18



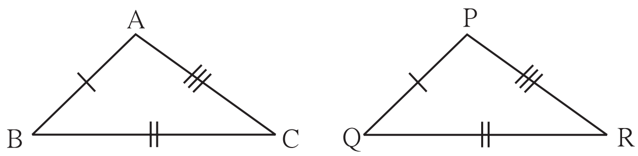
Remember this !

We have constructed triangles using the given information about parts of triangles. (For example, two angles and the included side, three sides, two sides and an included angle). We have experienced that the triangle constructed with any of these information is unique. So if by some one-to-one correspondence between two triangles, these three parts of one triangle are congruent with corresponding three parts of the other triangle then the two triangles are congruent. Then we come to know that in that correspondence their three angles and three sides are congruent. If two triangles are congruent then their respective angles and respective sides are congruent. This property is useful to solve many problems in Geometry.

Practice set 3.2

1. In each of the examples given below, a pair of triangles is shown. Equal parts of triangles in each pair are marked with the same signs. Observe the figures and state the test by which the triangles in each pair are congruent.

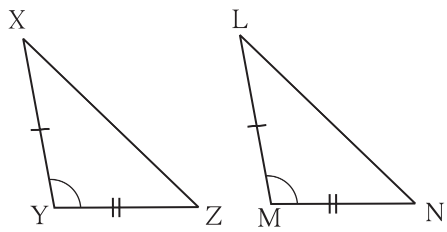
(i)



By . . . . test

Δ ABC $\cong$ Δ PQR

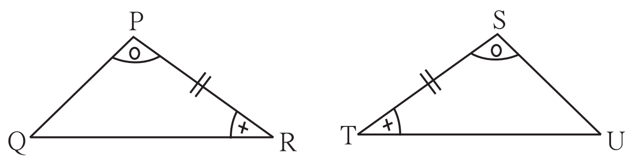
(ii)



By . . . . test

Δ XYZ $\cong$ Δ LMN

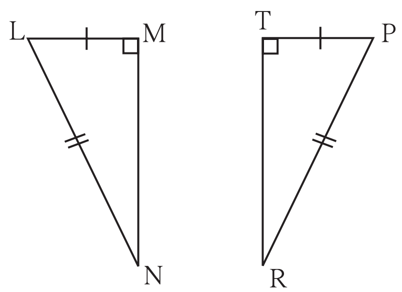
(iii)



By . . . . test

Δ PRQ $\cong$ Δ STU

(iv)



By . . . . test

Δ LMN $\cong$ Δ PTR

Fig. 3.19

2. Observe the information shown in pairs of triangles given below. State the test by which the two triangles are congruent. Write the remaining congruent parts of the triangles.

(i)

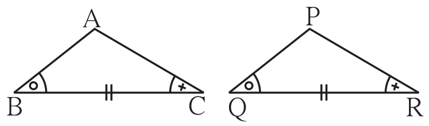


Fig. 3.20

From the information shown in the figure,

in Δ ABC and Δ PQR

∠ABC ≅ ∠PQR

seg BC ≅ seg QR

∠ACB ≅ ∠PRQ

∴ Δ ABC ≅ Δ PQR .... [ ] test

∴ ∠BAC ≅ [ ] ....corresponding angles of congruent triangles.

seg AB ≅ [ ]

and [ ] ≅ seg PR

} .... corresponding sides of congruent triangles

(ii)

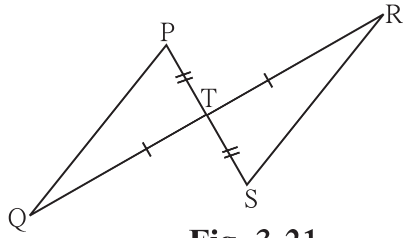


Fig. 3.21

From the information shown in the figure,,

In Δ PTQ and Δ STR

seg PT ≅ seg ST

∠PTQ ≅ ∠STR....vertically opposite angles

seg TQ ≅ seg TR

∴ Δ PTQ ≅ Δ STR .... [ ] test

∴ ∠TPQ ≅ [ ]

and [ ] ≅ ∠TRS

} ..... corresponding angles of congruent triangles.

seg PQ ≅ [ ] corresponding sides of congruent triangles.

3. From the information shown in the figure, state the test assuring the congruence of Δ ABC and Δ PQR. Write the remaining congruent parts of the triangles.

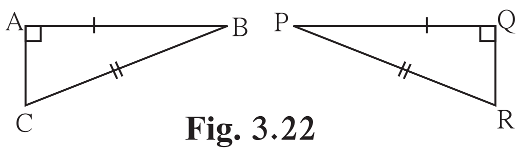


Fig. 3.22

4. As shown in the following figure, in Δ LMN and Δ PNM, LM = PN, LN = PM. Write the test which assures the congruence of the two triangles. Write their remaining congruent parts.

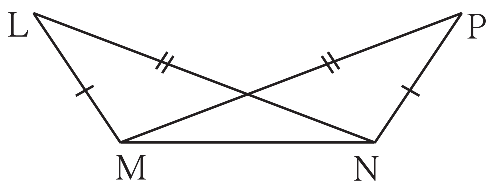


Fig. 3.23

5. In figure 3.24, seg AB ≅ seg CB and seg AD ≅ seg CD.

Prove that

Δ ABD ≅ Δ CBD

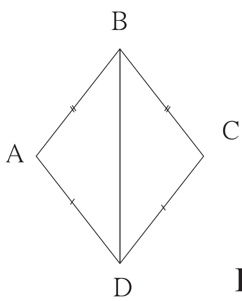


Fig. 3.24

Please note : corresponding sides of congruent triangles in short we write c.s.c.t. and corresponding angles of congruent triangles in short we write c.a.c.t.

6. In figure 3.25, ∠P ≅ ∠R
seg PQ ≅ seg RQ
Prove that,
Δ PQT ≅ Δ RQS

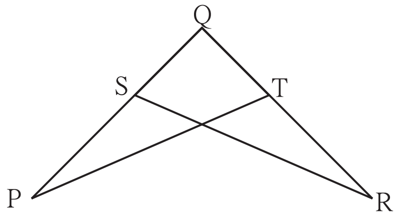
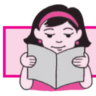


Fig. 3.25


Let's learn.

Isosceles triangle theorem

Theorem : If two sides of a triangle are congruent then the angles opposite to them are congruent.

Given : In Δ ABC, side AB ≅ side AC

To prove : ∠ABC ≅ ∠ACB

Construction : Draw the bisector of ∠BAC which intersects side BC at point D.

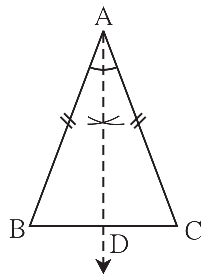


Fig. 3.26

Proof : In Δ ABD and Δ ACD

seg AB ≅ seg AC .... given

∠BAD ≅ ∠CAD........construction

seg AD ≅ seg AD .... common side

∴ Δ ABD ≅ Δ ACD ...... ______

∴ ∠ABD ≅ ______ .... (c.a.c.t.)

∴ ∠ABC ≅ ∠ACB ∵ B - D - C

Corollary : If all sides of a triangle are congruent then its all angles are congruent. (write the proof of this corollary.)

Converse of isosceles triangle theorem

Theorem : If two angles of a triangle are congruent then the sides opposite to them are congruent.

Given : In Δ PQR, ∠PQR ≅ ∠PRQ

To prove : Side PQ ≅ side PR

Construction : Draw the bisector of ∠P intersecting side QR at point M

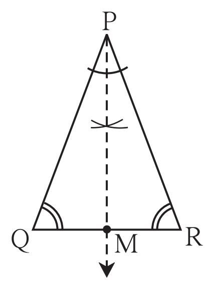


Fig. 3.27

Proof : In Δ PQM and Δ PRM

∠PQM ≅ ______........ given

∠QPM ≅ ∠RPM........ ______

seg PM ≅ ______ .... common side

∴ Δ PQM ≅ Δ PRM ...... ______ test

∴ seg PQ ≅ seg PR....... c.s.c.t.

Corollary : If three angles of a triangle are congruent then its three sides also are congruent. (Write the proof of this corollary yourself.)

Both the above theorems are converses of each other also.

Similarly the corollaries of the theorems are converses of each other.

Use your brain power!

(1) Can the theorem of isosceles triangle be proved doing a different construction ?

(2) Can the theorem of isosceles triangle be proved without doing any construction ?

Let's learn.

Property of 30° - 60° - 90° triangle

Activity I

Every student in the group should draw a right angled triangle, one of the angles measuring 30°. The choice of lengths of sides should be their own. Each one should measure the length of the hypotenuse and the length of the side opposite to 30° angle.

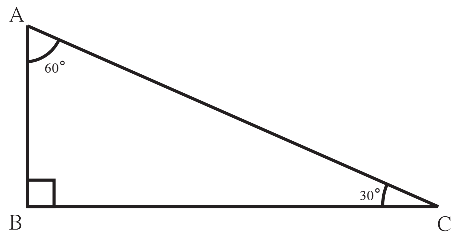


Fig. 3.28

One of the students in the group should fill in the following table.

Triangle Number	1	2	3	4
Length of the side opposite to 30° angle				
Length of the hypotenuse				

Did you notice any property of sides of right angled triangle with one of the angles measuring 30°?

Activity II

The measures of angles of a set square in your compass box are $30^\circ, 60^\circ$ and 90°. Verify the property of the sides of the set square.

Let us prove an important property revealed from these activities.

Theorem : If the acute angles of a right angled triangle have measures 30° and 60°, then the length of the side opposite to 30° angle is half the length of the hypotenuse.

(Fill in the blanks and complete the proof .)

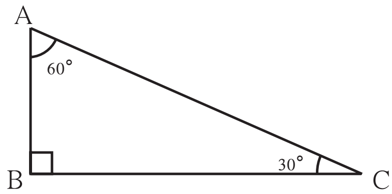


Fig. 3.29

Given : In Δ ABC

$\angle B = 90^\circ$, $\angle C = 30^\circ$, $\angle A = 60^\circ$

To prove : $AB = \frac{1}{2} AC$

Construction : Take a point D on the extended seg AB such that AB = BD. Draw seg DC.

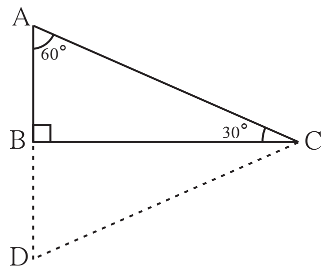


Fig. 3.30

Proof : Δ ABC and Δ DBC

seg AB $\cong$ seg DB .... [ ]

$\angle ABC \cong \angle DBC$ .... [ ]

seg BC $\cong$ seg BC .... [ ]

$\therefore$ $\Delta ABC \cong \Delta DBC$ ..... [ ]

$\therefore$ $\angle BAC \cong \angle BDC$ .... (c.a.c.t.)

In Δ ABC, $\angle BAC = 60^\circ$ $\therefore$ $\angle BDC = 60^\circ$

$\angle DAC = \angle ADC = \angle ACD = 60^\circ$ ...$\because$ sum of angles of Δ ADC is 180°

$\therefore$ Δ ADC is an equilateral triangle.

$\therefore$ AC = AD = DC .... corollary of converse of isosceles triangle theorem

But $AB = \frac{1}{2} AD$........ construction

$\therefore AB = \frac{1}{2} AC$ .... $\because$ AD = AC

Activity

With the help of the Figure 3.29 above fill in the blanks and complete the proof of the following theorem.

Theorem : If the acute angles of a right angled triangle have measures 30° and 60° then the length of the side opposite to 60° angle is $\frac{\sqrt{3}}{2} \times$ hypotenuse

Proof : In the above theorem we have proved $AB = \frac{1}{2} AC$

$AB^2 + BC^2 =$ [ ] .... Pythagoras theorem

$\frac{1}{4} AC^2 + BC^2 =$ [ ]

$\therefore BC^2 = AC^2 - \frac{1}{4} AC^2$

$\therefore BC^2 =$ [ ]

$\therefore BC = \frac{\sqrt{3}}{2} AC$

Activity : Complete the proof of the theorem.

Theorem : If measures of angles of a triangle are $45^\circ, 45^\circ, 90^\circ$ then the length of each side containing the right angle is $\frac{1}{\sqrt{2}} \times$ hypotenuse.

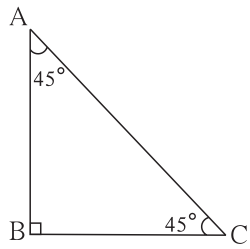


Fig. 3.31

Proof : In Δ ABC, $\angle B = 90^\circ$ and $\angle A = \angle C = 45^\circ$

$\therefore$ BC = AB

By Pythagoras theorem

$AB^2 + BC^2 = \square$

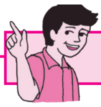
$AB^2 + \square = AC^2$ ... $\because$ (BC = AB)

$\therefore 2AB^2 = \square$

$\therefore AB^2 = \square$

$\therefore AB = \frac{1}{\sqrt{2}} AC$

This property is called 45°- 45°- 90° theorem.



Remember this !

(1) If the acute angles of a right angled triangle are 30°, 60° then the length of side opposite to 30° angle is half of hypotenuse and the length of side opposite to 60° angle is $\frac{\sqrt{3}}{2}$ hypotenuse . This property is called $30^\circ-60^\circ-90^\circ$ theorem.

(2) If acute angles of a right angled triangle are $45^\circ, 45^\circ$ then the length of each side containing the right angle is $\frac{\text{hypotenuse}}{\sqrt{2}}$.

This property is called $45^\circ-45^\circ-90^\circ$ theorem.

Let's recall.

Median of a triangle

The segment joining a vertex and the mid-point of the side opposite to it is called a **Median** of the triangle.

In Figure 3.32, point D is the mid point of side BC.

$\therefore$ seg AD is a median of Δ ABC.

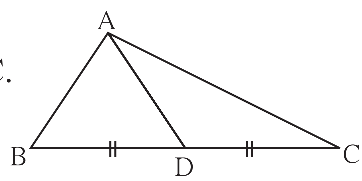


Fig. 3.32

Activity I : Draw a triangle ABC. Draw medians AD, BE and CF of the triangle. Let their point of concurrence be G, which is called the centroid of the triangle. Compare the lengths of AG and GD with a divider. Verify that the length of AG is twice the length of GD. Similarly, verify that the length of BG is twice the length of GE and the length of CG is twice the length of GF. Hence note the following property of medians of a triangle.

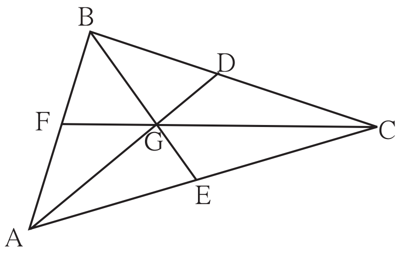


Fig. 3.33

The point of concurrence of medians of a triangle divides each median in the ratio 2 : 1.

Activity II : Draw a triangle ABC on a card board. Draw its medians and denote their point of concurrence as G. Cut out the triangle.

Now take a pencil. Try to balance the triangle on the flat tip of the pencil. The triangle is balanced only when the point G is on the flat tip of the pencil.

This activity shows an important property of the **centroid** (point of concurrence of the medians) of the triangle.

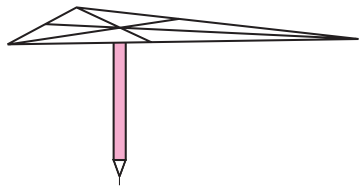
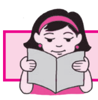


Fig. 3.34


Let's learn.

Property of median drawn on the hypotenuse of right triangle

Activity : In the figure 3.35, Δ ABC is a right angled triangle. seg BD is the median on hypotenuse.

Measure the lengths of the following segments.

AD =......... DC =............ BD = ...

From the measurements verify that $BD = \frac{1}{2} AC$.

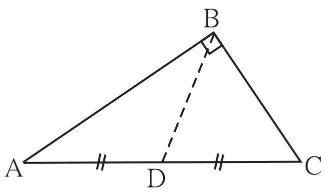


Fig. 3.35

Now let us prove the property, the length of the median is half the length of the hypotenuse.

Theorem : In a right angled triangle, the length of the median of the hypotenuse is half the length of the hypotenuse.

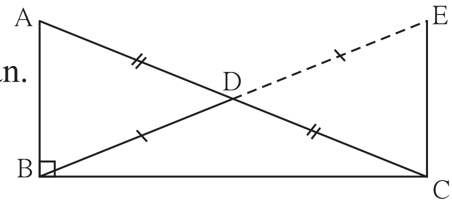


Fig. 3.36

Given : In Δ ABC, $\angle B = 90^\circ$, seg BD is the median.

To prove : $BD = \frac{1}{2} AC$

Construction : Take point E on the ray BD such that B - D - E and $l(BD) = l(DE)$. Draw seg EC.

Proof : (Main steps are given. Write the steps in between with reasons and complete the proof.)

Δ ADB $\cong$ Δ CDE .... by S-A-S test

line AB $\parallel$ line EC ....by test of alternate angles

Δ ABC $\cong$ Δ ECB .... by S-A-S test

$BD = \frac{1}{2}$ AC

Remember this !

In a right angled triangle, the length of the median on its hypotenuse is half the length of the hypotenuse.

Practice set 3.3

1. Find the values of x and y using the information shown in figure 3.37. Find the measure of $\angle ABD$ and $m\angle ACD$.

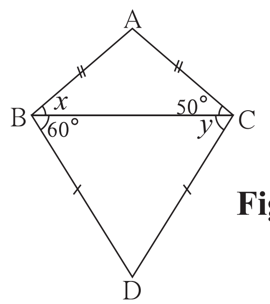


Fig. 3.37

2. The length of hypotenuse of a right angled triangle is 15. Find the length of median of its hypotenuse.

3. In Δ PQR, $\angle Q = 90^\circ$, PQ = 12, QR = 5 and QS is a median. Find $l(QS)$.

4. In figure 3.38, point G is the point of concurrence of the medians of Δ PQR . If GT = 2.5, find the lengths of PG and PT.

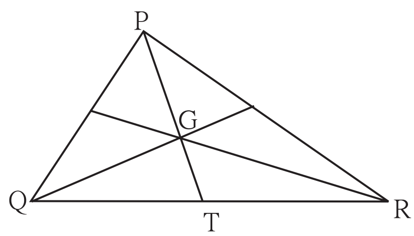


Fig. 3.38

Let's recall.

Activity : Draw a segment AB of convenient length. Lebel its mid-point as M. Draw a line *l* passing through the point M and perpendicular to seg AB.

Did you notice that the line *l* is the perpendicular bisector of seg AB ?

Now take a point P anywhere on line *l*. Compare the distance PA and PB with a divider. What did you find ? You should have noticed that PA = PB. This observation shows that any point on the perpendicular bisector of a segment is equidistant from its end points.

Now with the help of a compass take any two points like C and D, which are equidistant from A and B. Did all such points lie on the line *l* ? What did you notice from the observation ? Any point equidistant from the end points of a segment lies on the perpendicular bisector of the segment.

These two properties are two parts of the perpendicular bisector theorem. Let us now prove them.

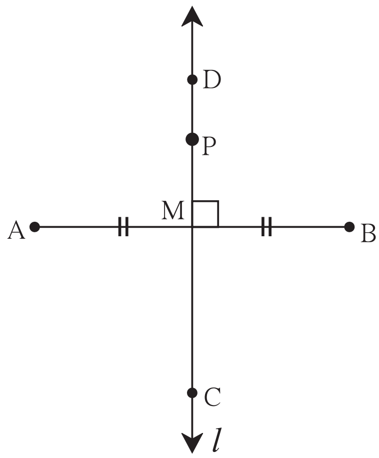


Fig. 3.39

Let's learn.

Perpendicular bisector theorem

Part I : Every point on the perpendicular bisector of a segment is equidistant from the end points of the segment.

Given : line *l* is the perpendicular bisector of seg AB at point M.

Point P is any point on *l*,

To prove : PA = PB

Construction : Draw seg AP and seg BP.

Proof : In Δ PMA and Δ PMB

seg PM ≅ seg PM .... common side

∠PMA ≅ ∠PMB ....each is a right angle

seg AM ≅ seg BM ....given

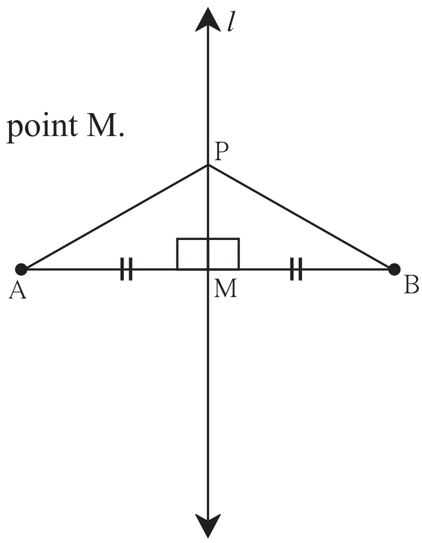


Fig. 3.40

$\therefore$ Δ PMA $\cong$ Δ PMB ...... S-A-S test

$\therefore$ seg PA $\cong$ seg PB ....c.s.c.t.

$\therefore$ l (PA) = l (PB)

Hence every point on the perpendicular bisector of a segment is equidistant from the end points of the segment.

Part II : Any point equidistant from the end points of a segment lies on the perpendicular bisector of the segment.

Given : Point P is any point equidistant from the end points of seg AB. That is, PA = PB.

To prove: Point P is on the perpendicular bisector of seg AB.

Construction : Take mid-point M of seg AB and draw line PM.

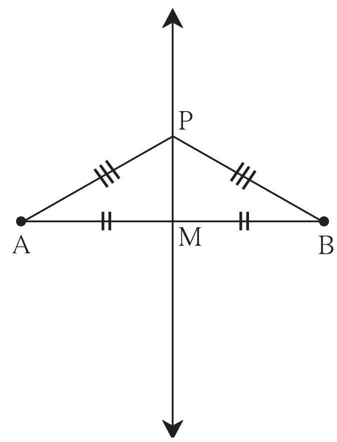


Fig. 3.41

Proof : In Δ PAM and Δ PBM

seg PA $\cong$ seg PB .... [ ]

seg AM $\cong$ seg BM .... [ ]

seg PM $\cong$ [ ] .... common side

$\therefore$ Δ PAM $\cong$ Δ PBM ...... [ ] test.

$\therefore$ $\angle$PMA $\cong$ $\angle$PMB.......c.a.c.t.

But $\angle$PMA + [ ] = 180°

$\angle$PMA + $\angle$PMA = 180° .... ($\because$ $\angle$PMB = $\angle$PMA)

2 $\angle$PMA = [ ]

$\therefore$ $\angle$PMA = 90°

$\therefore$ seg PM $\perp$ seg AB ......(1)

But Point M is the midpoint of seg AB. ......construction .... (2)

$\therefore$ line PM is the perpendicular bisector of seg AB. So point P is on the perpendicular bisector of seg AB.

Angle bisector theorem

Part I : Every point on the bisector of an angle is equidistant from the sides of the angle.

Given : Ray QS is the bisector of $\angle$PQR .
Point A is any point on ray QS
seg AB $\perp$ ray QP seg AC $\perp$ ray QR

To prove : seg AB $\cong$ seg AC

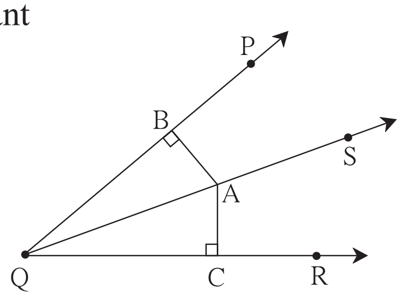


Fig. 3.42

Proof : Write the proof using test of congruence of triangles.

Part II : Any point equidistant from sides of an angle is on the bisector of the angle.

Given : A is a point in the interior of $\angle PQR$.

seg AC $\perp$ ray QR seg AB $\perp$ ray QP

and AB = AC

To prove : Ray QA is the bisector of $\angle PQR$.

That is $\angle BQA = \angle CQA$

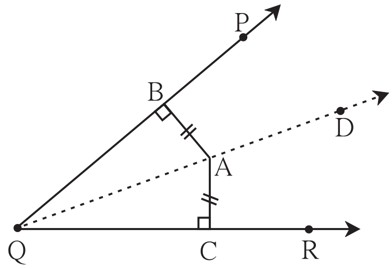


Fig. 3.43

Proof : Write the proof using proper test of congruence of triangles.

Let's recall.

Activity

As shown in the figure, draw Δ XYZ such that XZ > side XY

Find which of $\angle Z$ and $\angle Y$ is greater.

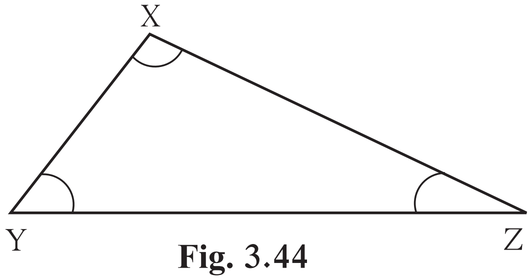
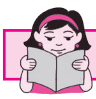


Fig. 3.44



Let's learn.

Properties of inequalities of sides and angles of a triangle

Theorem : If two sides of a triangle are unequal, then the angle opposite to the greater side is greater than angle opposite to the smaller side.

Given : In Δ XYZ, side XZ > side XY

To prove : $\angle XYZ > \angle XZY$

Construction : Take point P on side XZ such that XY = XP, Draw seg YP.

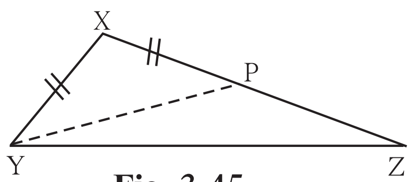


Fig. 3.45

Proof : In Δ XYP

XY = XP ....construction

$\therefore$ $\angle XYP = \angle XPY$.....isosceles triangle theorem .....(I)

$\angle XPY$ is an exterior angle of Δ YPZ.

$\therefore$ $\angle XPY > \angle PZY$ ....exterior angle theorem

$\therefore$ $\angle XYP > \angle PZY$ ....from (I)

$\therefore$ $\angle XYP + \angle PYZ > \angle PZY$ ....If $a > b$ and $c > 0$ then $a + c > b$

$\therefore$ $\angle XYZ > \angle PZY$, that is $\angle XYZ > \angle XZY$

Theorem : If two angles of a triangle are unequal then the side opposite to the greater angle is greater than the side opposite to smaller angle.

The theorem can be proved by indirect proof. Complete the following proof by filling in the blanks.

Given : In ΔABC, $\angle B > \angle C$

To prove : $AC > AB$

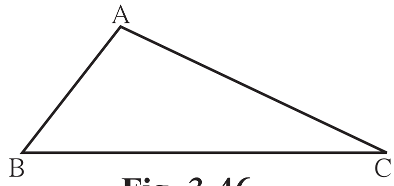


Fig. 3.46

Proof : There are only three possibilities regarding the lengths of side AB and side AC of Δ ABC.

(i) $AC < AB$ (ii) ______ (iii) ______

(i) Let us assume that $AC < AB$.

If two sides of a triangle are unequal then the angle opposite to greater side is ______ .

$\therefore$ $\angle C >$ ______

But $\angle C < \angle B$ .... (given)

This creates a contradiction.

$\therefore$ ______ < ______ is wrong.

(ii) If $AC = AB$

then $\angle B = \angle C$

But ______ > ______ ...... (given)

This also creates a contradiction.

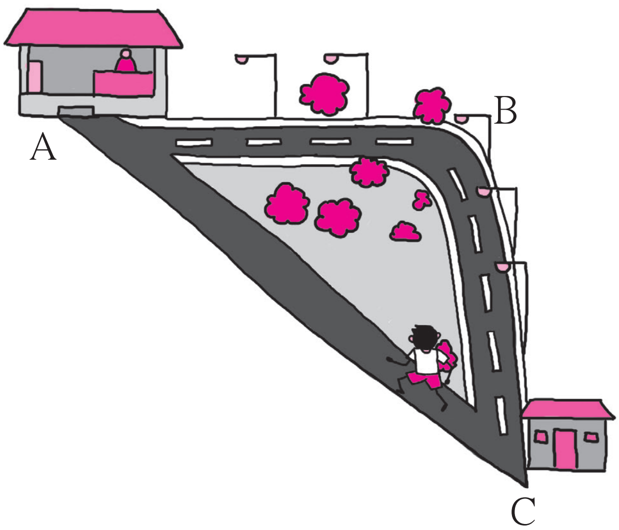
$\therefore$ ______ = ______ is wrong

$\therefore$ $AC > AB$ is the only remaining possibility.

$\therefore$ $AC > AB$

Let's recall.

As shown in the adjacent picture, there is a shop at A. Sameer was standing at C. To reach the shop, he choose the way $C \rightarrow A$ instead of $C \rightarrow B \rightarrow A$, because he knew that the way $C \rightarrow A$ was shorter than the way $C \rightarrow B \rightarrow A$. So which property of a triangle had he realised ?



The sum of two sides of a triangle is greater than its third side.

Let us now prove the property.

Theorem : The sum of any two sides of a triangle is greater than the third side.

Given : Δ ABC is any triangle.

To prove : AB + AC > BC
AB + BC > AC
AC + BC > AB

Construction : Take a point D on ray BA such that AD = AC.

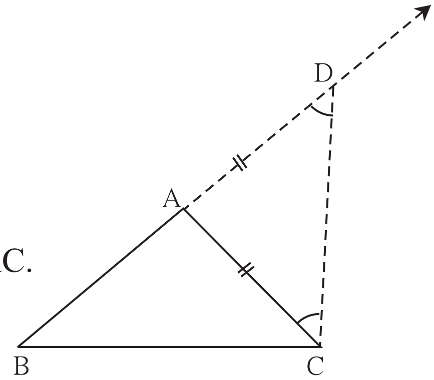


Fig. 3.47

Proof : In Δ ACD, AC = AD ..... construction

$\therefore$ $\angle$ACD = $\angle$ADC ...... c.a.c.t.

$\therefore$ $\angle$ACD + $\angle$ACB > $\angle$ADC

$\therefore$ $\angle$BCD > $\angle$ADC

$\therefore$ side BD > side BC ....the side opposite to greater angle is greater

$\therefore$ BA + AD > BC ....$\because$ BD = BA + AD

BA + AC > BC .... $\because$ AD = AC

Similarly we can prove that AB + BC > AC

and BC + AC > AB.

Practice set 3.4

1. In figure 3.48, point A is on the bisector of $\angle$XYZ.
If AX = 2 cm then find AZ.

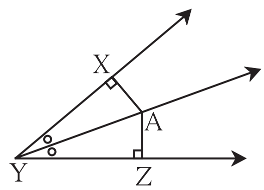


Fig. 3.48

2. In figure 3.49, $\angle$RST = 56°, seg PT $\perp$ ray ST, seg PR $\perp$ ray SR and seg PR $\cong$ seg PT
Find the measure of $\angle$RSP.
State the reason for your answer.

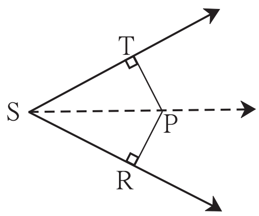


Fig. 3.49

3. In Δ PQR, PQ = 10 cm, QR = 12 cm, PR = 8 cm. Find out the greatest and the smallest angle of the triangle.

4. In Δ FAN, $\angle$F = 80°, $\angle$A = 40° . Find out the greatest and the smallest side of the triangle. State the reason.

5. Prove that an equilateral triangle is equiangular.

6. Prove that, if the bisector of ∠BAC of Δ ABC is perpendicular to side BC, then Δ ABC is an isosceles triangle.

7. In figure 3.50, if seg PR ≅ seg PQ, show that seg PS > seg PQ.

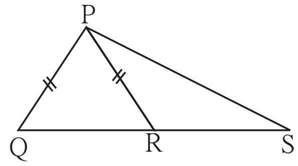


Fig. 3.50

8. In figure 3.51, in Δ ABC, seg AD and seg BE are altitudes and AE = BD.
Prove that seg AD ≅ seg BE

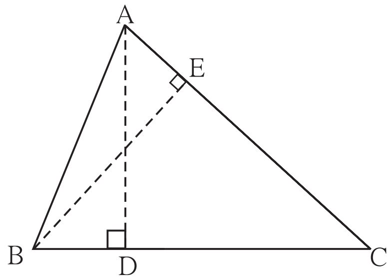
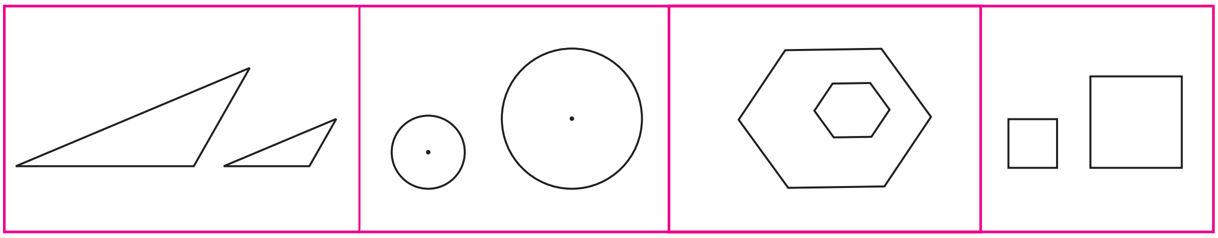


Fig. 3.51

Let's learn.

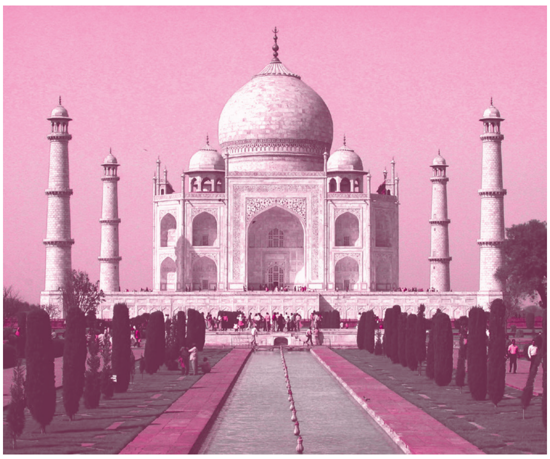
Similar triangles

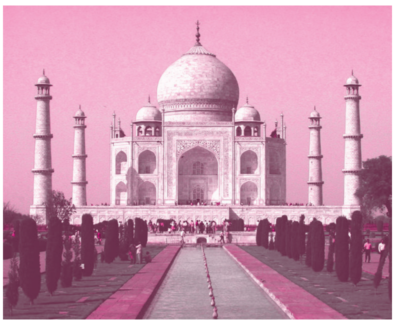
Observe the following figures.



The pairs of figures shown in each part have the same shape but their sizes are different. It means that they are not congruent.

Such like looking figures are called similar figures.





We find similarity in a photo and its enlargement, also we find similarity between a road-map and the roads.

The proportionality of all sides is an important property of similarity of two figures. But the angles in the figures have to be of the same measure. If the angle between this roads is not the same in its map, then the map will be misleading.

ICT Tools or Links

Take a photograph on a mobile or a computer. Recall what you do to reduce it or to enlarge it. Also recall what you do to see a part of the photograph in detail.

Now we shall learn properties of similar triangles through an activity.

Activity : On a card-sheet, draw a triangle of sides 4 cm, 3 cm and 2 cm. Cut it out. Make 13 more copies of the triangle and cut them out from the card sheet.
Note that all these triangular pieces are congruent. Arrange them as shown in the following figure and make three triangles out of them.

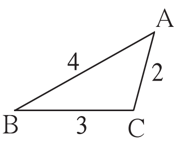


Fig. 3.52

1 triangle

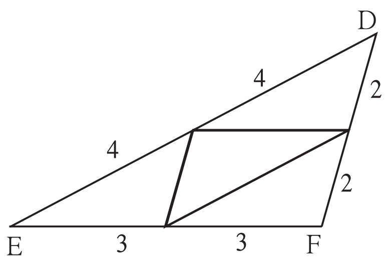


Fig. 3.53

4 triangles

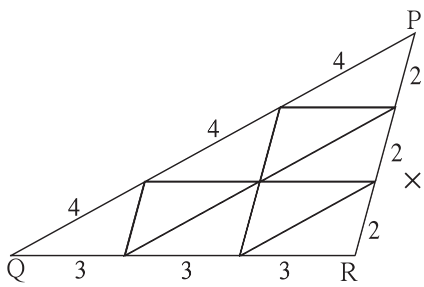


Fig. 3.54

9 triangles

Δ ABC and Δ DEF are similar in the correspondence ABC $\leftrightarrow$ DEF.

$\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$

and $\frac{AB}{DE} = \frac{4}{8} = \frac{1}{2}$; $\frac{BC}{EF} = \frac{3}{6} = \frac{1}{2}$; $\frac{AC}{DF} = \frac{2}{4} = \frac{1}{2}$,

........the corresponding sides are in proportion.

Similarly, consider Δ DEF and Δ PQR. Are their angles congruent and sides proportional in the correspondence DEF $\leftrightarrow$ PQR ?

Let's learn.

Similarity of triangles

In Δ ABC and Δ PQR, If (i) $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$ and

(ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$; then Δ ABC and Δ PQR are called similar triangles.

'Δ ABC and Δ PQR are similar' is written as '$\Delta ABC \sim \Delta PQR$'.

Let us learn the relation between the corresponding angles and corresponding sides of similar triangles through an activity.

Activity : Draw a triangle $\Delta A_1B_1C_1$ on a card-sheet and cut it out. Measure $\angle A_1, \angle B_1, \angle C_1$.

Draw two more triangles $\Delta A_2B_2C_2$ and $\Delta A_3B_3C_3$ such that

$\angle A_1 = \angle A_2 = \angle A_3$, $\angle B_1 = \angle B_2 = \angle B_3$, $\angle C_1 = \angle C_2 = \angle C_3$

and $B_1C_1 > B_2C_2 > B_3C_3$. Now cut these two triangles also. Measure the lengths of the three triangles. Arrange the triangles in two ways as shown in the figure.

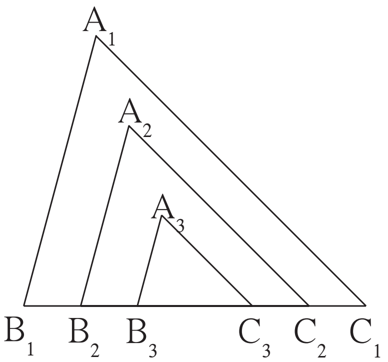


Fig. 3.55

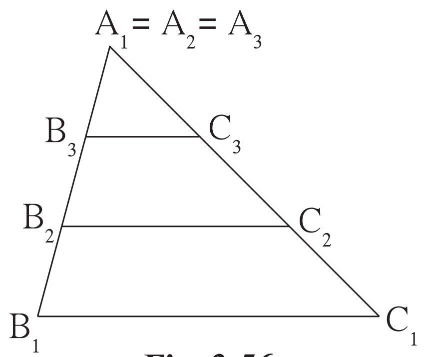


Fig. 3.56

Check the ratios $\frac{A_1B_1}{A_2B_2}, \frac{B_1C_1}{B_2C_2}, \frac{A_1C_1}{A_2C_2}$. You will notice that the ratios are equal.

Similarly, see whether the ratios $\frac{A_1C_1}{A_3C_3}, \frac{B_1C_1}{B_3C_3}, \frac{A_1B_1}{A_3B_3}$ are equal.

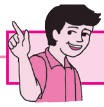
From this activity note that, when corresponding angles of two triangles are equal, the ratios of their corresponding sides are also equal. That is, their corresponding sides are in the same proportion.

We have seen that, in Δ ABC and Δ PQR if

(i) $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, then (ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

This means, if corresponding angles of two triangles are equal then the corresponding sides are in the same proportion.

This rule can be proved elaborately. We shall use it to solve problems.



Remember this !

- If corresponding angles of two triangles are equal then the two triangles are similar.
- If two triangles are similar then their corresponding sides are in proportion and corresponding angles are congruent.

Ex. Some information is shown in Δ ABC and Δ PQR in figure 3.57. Observe it. Hence find the lengths of side AC and PQ.

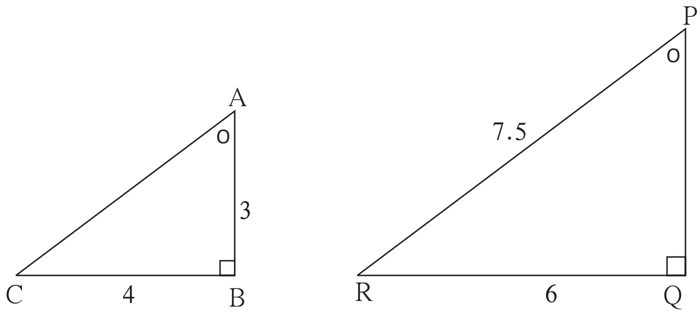


Fig. 3.57

Solution : The sum of all angles of a triangle is 180°.

It is given that,

$\angle A = \angle P$ and $\angle B = \angle Q$ $\therefore$ $\angle C = \angle R$

$\therefore$ Δ ABC and Δ PQR are equiangular triangles.

$\therefore$ there sides are propotional.

$$\therefore \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

$$\therefore \frac{3}{PQ} = \frac{4}{6} = \frac{AC}{7.5}$$

$$\therefore 4 \times PQ = 18$$

$$\therefore PQ = \frac{18}{4} = 4.5$$

Similarly $6 \times AC = 7.5 \times 4$

$$\therefore AC = \frac{7.5 \times 4}{6} = \frac{30}{6} = 5$$

Practice set 3.5

1. If Δ XYZ ~ Δ LMN, write the corresponding angles of the two triangles and also write the ratios of corresponding sides.
2. In Δ XYZ, XY = 4 cm, YZ = 6 cm, XZ = 5 cm, If Δ XYZ ~ Δ PQR and PQ = 8 cm then find the lengths of remaining sides of Δ PQR.
3. Draw a sketch of a pair of similar triangles. Label them. Show their corresponding angles by the same signs. Show the lengths of corresponding sides by numbers in proportion.

Let's recall.

While preparing a map of a locality, you have to show the distances between different spots on roads with a proper scale. For example, 1 cm = 100 m, 1 cm = 50 m etc. Did you think of the properties of triangle ? Keep in mind that side opposite to greater angle is greater.

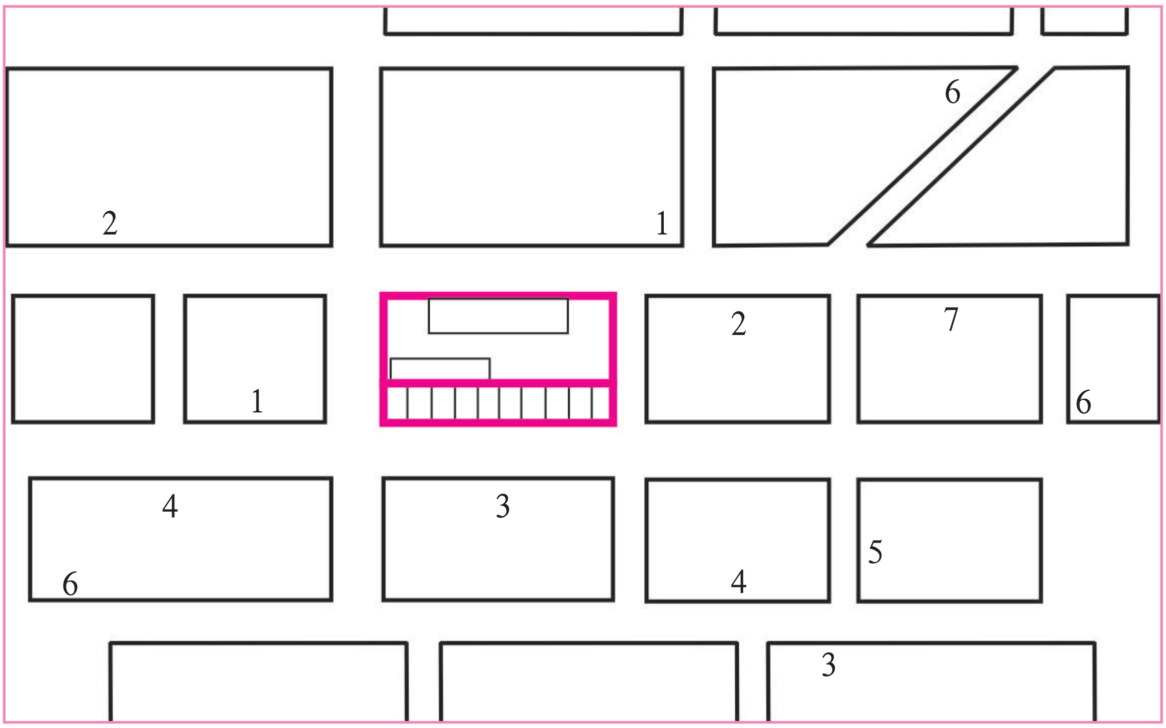
Project :

Prepare a map of road surrounding your school or home, upto a distance of about 500 metre.

How will you measure the distance between two spots on a road ?

While walking, count how many steps cover a distance of about two metre. Suppose, your three steps cover a distance of 2 metre. Considering this proportion 90 steps means 60 metre. In this way you can judge the distances between different spots on roads and also the lengths of roads. You have to judge the measures of angles also where two roads meet each other. Choosing a proper scale for lengths of roads, prepare a map. Try to show shops, buildings, bus stops, rickshaw stand etc. in the map.

A sample map with legend is given below.



Legend: 1. Book store 2. Bus stop 3. Stationery shop 4. Bank
5. Medical store 6. Restaurant 7. Cycle shop

Problem set 3

1. Choose the correct alternative answer for the following questions.

   (i) If two sides of a triangle are 5 cm and 1.5 cm, the lenght of its third side cannot be . . . .

   (A) 3.7 cm (B) 4.1 cm (C) 3.8 cm (D) 3.4 cm

   (ii) In Δ PQR, If $\angle R > \angle Q$ then . . . .

   (A) $QR > PR$ (B) $PQ > PR$ (C) $PQ < PR$ (D) $QR < PR$

   (iii) In Δ TPQ, $\angle T = 65°$, $\angle P = 95°$ which of the following is a true statement ?

   (A) $PQ < TP$ (B) $PQ < TQ$ (C) $TQ < TP < PQ$ (D) $PQ < TP < TQ$

2. Δ ABC is isosceles in which AB = AC. Seg BD and seg CE are medians. Show that BD = CE.

3. In Δ PQR, If $PQ > PR$ and bisectors of $\angle Q$ and $\angle R$ intersect at S. Show that $SQ > SR$.

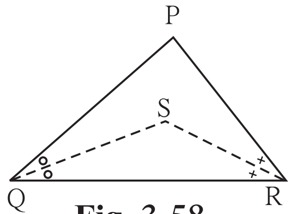


Fig. 3.58

4. In figure 3.59, point D and E are on side BC of Δ ABC, such that BD = CE and AD = AE. Show that $\Delta ABD \cong \Delta ACE$.

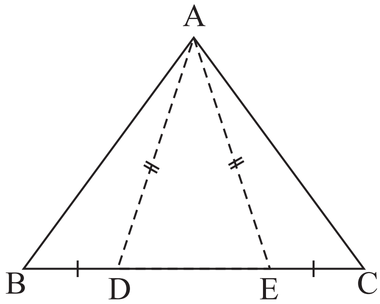


Fig. 3.59

5. In figure 3.60, point S is any point on side QR of Δ PQR. Prove that : $PQ + QR + RP > 2PS$

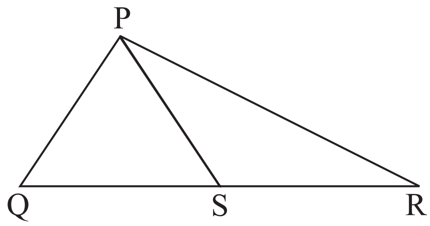


Fig. 3.60

6. In figure 3.61, bisector of $\angle BAC$ intersects side BC at point D. Prove that AB > BD

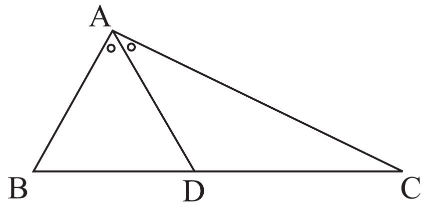


Fig. 3.61

7. In figure 3.62, seg PT is the bisector of $\angle QPR$. A line parallel to seg PT and passing through R intersects ray QP at point S. Prove that PS = PR.

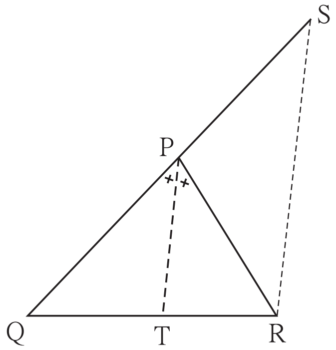


Fig. 3.62

8. In figure 3.63, seg AD $\perp$ seg BC. seg AE is the bisector of $\angle CAB$ and C - E - D. Prove that

$\angle DAE = \frac{1}{2} (\angle C - \angle B)$

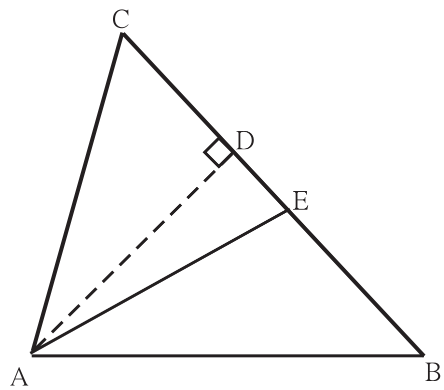


Fig. 3.63

Use your brain power!

We have learnt that if two triangles are equiangular then their sides are in proportion. What do you think if two quadrilaterals are equiangular ? Are their sides in proportion? Draw different figures and verify.

Verify the same for other polygons.